Further investigations:
Show your student graphs in newspapers, journals, or on the Internet. Identify the domains and ranges and discuss whether they represent discrete or continuous data. When watching television with your student, pick statements from the commercials and restate them as conditional statements. Then state the converse, inverse, and contrapositive. Evaluate the truth value of each statement.

Look for sequences in your world such as hours worked or number of seats at a theater. Ask your student to represent them recursively, in closed form, and in function notation.

Terminology:
Conclusion: In a conditional statement, the part that follows "then."
Contrapositive: A conditional statement that negates and reverses the hypothesis and the conclusion.
Converse: A conditional statement that reverses the hypothesis and the conclusion.
Continuous: A set of data that can include any Real-numbered value in a given interval such as temperature, time, and length.
Discrete: A set of data that represents a situation where the possibilities are distinct and separated from each other such as counts of people.
Domain: The set of all possible values for the independent or input variable in a function.

Hypothesis: In a conditional statement, the part that follows "if."
Inverse: A conditional statement that negates the hypothesis and the conclusion.
Range: The set of all possible values for the dependent or output variable in a function.

Clues:
To a large extent, applied mathematics consists of modeling various phenomena by functions, using mathematics to analyze these functions, and then using this mathematical analysis to obtain insight into the phenomena. We can model more and more things if we have a larger repertoire of functions.

Book’em:
Through the Looking Glass by Lewis Carroll

Related Files: www.ceismc.gatech.edu/csi

Function Families

Students will: Math I - 1 of 6
• Explore properties of basic quadratic, cubic, absolute value, square root, and rational functions
• Determine the range given the domain and rule of correspondence for a function
• Represent functions with function notation and use the notation to ask and answer questions about relationships
• Read and draw graphs of functional relationships
• Recognize and evaluate logical relationships between a statement and its converse, its inverse, and its contrapositive.

Classroom Cases:
1. Ina has a job after school delivering papers. She is paid $5 per week plus $.10 for each paper delivered. Make a table and a graph to show the relationship between the number of papers delivered and amount earned each week. Write a formula in function notation to represent the relationship. What is the domain? What is the range?

Case Closed - Evidence:

<table>
<thead>
<tr>
<th>Number of papers delivered, n</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weekly earnings, E(n)</td>
<td>6</td>
<td>7</td>
<td>7.50</td>
<td>8</td>
<td>8.20</td>
</tr>
</tbody>
</table>

Ina could earn in a week. It can be represented \( \{ n \in W \mid n \geq 0 \} \). The range includes all the amounts Ina could earn in a week. It can be represented \( \{ E(n) \in Q \mid E(n) \geq 0 \} \). Since the number of papers must be whole numbers, the points on the graph should not be connected.

2. Write the sentences below in "if-then" form. Give the converse of each statement and tell whether its truth value is true or false.
   - If I am at a wedding, then I cry. If I cry, then I am at a wedding.
   - A rectangle is a quadrilateral with 4 right angles.
   - \( f(9) = 3 \) when \( f(x) = \sqrt{x} \)

Case Closed - Evidence:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Converse</th>
<th>Truth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>If I am at a wedding, then I cry.</td>
<td>If I cry, then I am at a wedding.</td>
<td>False. I also cry at the movies.</td>
</tr>
<tr>
<td>If a quadrilateral is a rectangle, then it has 4 right angles.</td>
<td>If a quadrilateral has 4 right angles, then it is a rectangle.</td>
<td>True</td>
</tr>
<tr>
<td>If ( f(x) = \sqrt{x} ), then ( f(9) = 3 )</td>
<td>If ( f(9) = 3 ), then ( f(x) = \sqrt{x} )</td>
<td>False. ( f(x) ) could be ( \sqrt{27} ).</td>
</tr>
</tbody>
</table>

3. Uncle Hank is building a shop. He needs a floor space of 1200 square feet. Make a table to show some of the possible lengths and widths for the shop. Draw a graph to show the relationship between width and length and represent the relationship in function notation. Write a function rule to calculate the length of the floor for any given width. Use your rule to determine the length of the floor when the width is 28 feet.

Case Closed - Evidence:

<table>
<thead>
<tr>
<th>Width</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>120</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>50</td>
<td>24</td>
</tr>
<tr>
<td>60</td>
<td>20</td>
</tr>
<tr>
<td>80</td>
<td>15</td>
</tr>
<tr>
<td>100</td>
<td>12</td>
</tr>
</tbody>
</table>

Let \( w = \) width and \( L(w) = \) length. Then \( L(w) = 1200/w \). If \( w = 28 \), then \( L(28) = 1200/28 \) or 42.6/7 feet.
Further investigations:

Involving your student with projects. Ask her to represent measures in terms of other measures. For example, if you are building a bookcase with a depth of $d$, its width might be $4d + 2$ and its height might be $6d - 7$. How much floor space would it occupy in terms of $d$?

Challenge your student with number puzzles such as “I am thinking of 2 numbers whose sum is -15 and whose product is 56. What are the 2 numbers?” [Answer: -7 and -8]

Invite your student to create a similar puzzle to stump you.

Show your student graphs in newspapers, journals, or on the Internet. Which ones look to be linear? Quadratic? Rational? Square Root?

**Algebra Investigations**

**Students will:**

- Add, subtract, multiply, and divide elementary polynomial, rational, and radical expressions
- Represent situations and relationships with algebraic expressions and equations
- Interpret algebraic expressions and equations
- Verify algebraic equivalence
- Test conjectures about relationships between operations on Real numbers and provide algebraic or geometric explanations or counterexamples

**Classroom Cases:**

1. Amy is making a picture frame with 1-inch wide mosaic tile. The length of the picture is two inches greater than the width. Write an expression for the number of one inch tiles needed for a picture with a width of $w$ inches. Write a different but equivalent expression for the number of mosaic tiles. Explain why your expressions are equivalent.

   **Case Closed - Evidence:**
   
   The area of the picture and the frame — area of the picture $(w + 2)(w + 4) - w(w + 2) = w^2 + 6w + 8 - w^2 - 2w$ Sum of areas of eight rectangles in the frame $1 + w + 1 + (w + 2) + 1 + w + 1 + (w + 2)$ The expressions are equivalent because when they are simplified, they both equal $4w + 8$.

2. U-Pac makes different size boxes. They vary the lengths of the edges of their cubical boxes by 3 inches. If the smallest box has edges that are $x$ inches long, write an expression to represent the area of the bottom of the next larger box. What will be the volume of that box?

   **Case Closed - Evidence:**
   
   The next larger box has edges that are $x + 3$ inches. The area of the box bottom will be $(x + 3)^2$ or $x^2 + 6x + 9$ square inches. The volume of this box will be $(x + 3)^3$ or $x^3 + 9x^2 + 27x + 27$ cubic inches.

3. a. Micah is practicing for the Junior Tour de Georgia bicycle race. He rode 35 miles in two hours, 30 minutes. When it started raining, he returned home, but the 35 miles took him 5 hours. What was his average rate of speed for the 70 miles?

   **Case Closed - Evidence:**
   
   The first part of the trip took $20/m$ hours and the second part took $15/n$. So, the total trip took $(20/m + 15/n)$ hours. His average speed will be the total distance divided by the total time:
   
   
   If they want the path to be as short as possible, then they should make the garden a square.
Further investigations:
Point out triangles and quadrilaterals in architecture. Ask your student to describe their properties. Encourage her to relate ideas from the classroom using the shapes found in buildings and other structures.
If your student is a member of a service club, suggest that the club make toys or mobiles that use the centroid as a balance point.

Geometry

Students will: Math 1 - 3 of 6
• Prove conjectures through multiple forms of justification
• Explore angles, triangle inequalities, congruencies, and points of concurrency
• Apply properties to determine special quadrilaterals

Classroom Cases:
1. If the distance from point $O$ to point $A$ is the same as the distance from point $O$ to point $B$, show that point $O$ is on the perpendicular bisector of segment $AB$. How could you use this proof to support other ideas in geometry?

Case Closed - Evidence:
I drew $AB$ and found its midpoint which I labeled $P$.
I drew $PO$, $PA$, and $BO$ to form two triangles: $\triangle AOP$ and $\triangle BOP$. These triangles are congruent because their corresponding sides are congruent. $AP = BP$ (since $P$ is the midpoint of $AB$), $AO = BO$ (information given in the problem), and $OP = OP$. Congruent triangles have congruent corresponding angles. So, $\angle OPA = \angle OPB$. But together these angles form a linear pair, and the sum of their measures is 180°. Since the two angles have equal measures, each measure is $\frac{1}{2}$ of 180° or 90°. Since $AB$ and $OP$ meet to form 90° angles, they are perpendicular, and since $P$ is the midpoint of $AB$, $OP$ is the perpendicular bisector of $AB$. Since $O$ is a point on $OP$, $O$ is a point on the perpendicular bisector of $AB$.

This proof is useful in locating the circumcenter of a triangle. To find the circumcenter you have to find a point that is equidistant from the vertices of a triangle. It turns out that this point will be the intersection of the perpendicular bisectors of the triangle’s sides.

2. Given the figure at right where $O$ is the center of the circle and $A$, $B$, and $C$ are points on the circle, show that $m \angle BOC = 2 \cdot m \angle BAC$. Then suggest what the figure might represent in the work world.

Case Closed - Evidence:

First, I drew the radius $\overline{AO}$. This formed two isosceles triangles (\(\triangle ABO\) and \(\triangle ACO\)) because in each triangle, two sides are radii of the same circle. Base angles of isosceles triangles are congruent so \(\angle ABO \cong \angle BAO\) and \(\angle ACO \cong \angle CAO\). Let \(x = m \angle BAO\) and \(y = m \angle CAO\). Then \(m \angle AOB = 180° - 2x\) and \(m \angle AOC = 180° - 2y\) because there are 180° in a triangle and two of the angles have the same measures. Then \(m \angle BOC = 360° - (180° - 2x) - (180° - 2y) = 2x + 2y = 2(x + y)\). But \(x + y = m \angle BAC\). Substituting again, \(m \angle BOC = 2(x + y) = 2m \angle BAC\). I think this figure could be a logo for a space research company or maybe a specially-touched machine part.

3. Marla works for a tool and die company. She received an order for a triangular part to fit a tractor. The order was smudged and Marla can read only two of the side lengths clearly: 7 inches and 11 inches. The smudged third side length could be 4 inches, 14 inches, or 19 inches. Which of these lengths is possible?

Case Closed - Evidence:
The sum of the lengths of any two sides of a triangle is greater than the length of the third side. But 7 + 11 < 19, and 4 + 7 = 11. The third side must be 14 inches because 7 + 11 > 14.

Book’em:
Flatland by Edwin A. Abbott

Related Files:
www.ceismc.gatech.edu/csi
Further investigations: Find baseball statistics for your student’s favorite team. Ask her to calculate the probability that the catcher will get at least 2 hits in his next 3 at-bats if he got a hit on the first at-bat. Repeat for other players.

If you play the Lottery, explain to your student what boxed and straight mean and ask him to calculate the expected winnings each way for playing “Cash 3” 10 times.

Brought to you by

Terminology:

Combination: A collection of objects in which the order of objects does not matter.

Conditional probability: The chance that a second event will occur given that another event has occurred.

Dependent events: Two events in which the outcome of one affects the probability of the other.

Expected value: A predicted value for a distribution based on the probability and value of the events in the distribution.

Factorial: The product of all the integers from 1 up to the integer in question. 5! = 1•2•3•4•5 = 120

Mean absolute deviation: Average of the absolute values of the deviations.

Mutually exclusive events: Two events that have no outcomes in common.

Outcome: A possible result.

Parameter: A numerical value that describes the population.

Permutation: Each different arrangement of a set of objects.

Summary statistics: Measures of the center (mean, median) and of the spread (quartiles, interquartile range) of a set of data.

Variability: A measure of the spread or dispersion of a data distribution.

Clues: Probability started from the analysis of games, and this is still the usual start in courses on the subject because games provide a context for probability that is not complicated by other issues. Nonetheless, probability is omnipresent in modern thought. Any kind of decision making under uncertainty has the basic ideas of probability lurking somewhere.

Book’em: Conned Again, Watson! by Colin Bruce

The Chance of Winning

Students will:

- Calculate probabilities based on angles and area models
- Compute simple permutations and combinations
- Calculate, display, and interpret summary statistics
- Calculate and interpret expected values
- Use simulations and statistics as tools to answer probability questions.

Classroom Cases:

1. A traveling carnival has a spinner game like the one shown. Based on the spinner, calculate the following:
   a. What is the probability of obtaining $1 on the first spin?
   b. Are you equally likely to land on $1 and $6?
   c. What are your chances of obtaining at least $5 on one spin?
   d. If you spin twice, what is the likelihood that you will have a sum of at most $4?
   e. Given that you landed on $1 on your first spin, what is the probability that the sum of your two spins will be at most $4?
   f. If you spin the spinner once, how much money would you expect to receive? How much money would you be willing to pay to play this game?

Case Closed - Evidence:

a. The spinner has 8 sections but they are not the same size. The larger ones are twice as big as the smaller sections. Since $1 is in a larger section, its area represents 2/12 of the circle. The probability of obtaining $1 then is 2/12.

b. No. $1 and $6 are in different size sections so their probabilities are different.

c. $12/12 + $2/12 + $3/12 = 2/12 + 1/12 + 1/12 = 6/12 = 1/2.

d. $12/12 + $2/12 + $3/12 = 2/12 + 1/12 + 1/12 = 6/12 = 1/2.

e. Given that you landed on $1 on your first spin reduces the sample space to $1, $2, $3, $4, $5, $6, $7, $8$.

f. The expected value would be $1(2/12) + $2(1/12) + $3(1/12) + $4(2/12) + $5(2/12) + $6(1/12) + $7(1/12) + $8(2/12) = $4.50. Since I only expect to win $4.50 on 1 spin, I would not be willing to risk more than that amount to play.

2. Suppose you take the driver’s permit test and randomly guess the answers. If there are 30 questions with 5 answer choices per question, what is the probability that you just pass the test (make exactly 70%)?

Class Closed - Evidence:

Since there are 3 choices for each question, the probability of guessing the correct answer on one question is 1/3 and the probability of missing that answer is 2/3. To make a passing grade, I must answer correctly 70% of 30 questions or 21 questions. The probability for making a 70 by just guessing will be 14307150 (1/3)21 (2/3)9 = 4.03 x 10-9.

3. The star player on your basketball team has an 85% free throw percentage and is in a one-and-one situation 200 times during the season. How many times can she be expected to score zero points? One point? Two points?

Case Closed - Evidence:

As the area model shows, the probability of her making the first shot and the second shot would be .85 • .85 = .7225. That would happen about 145 times. The probability that she makes the first shot and misses the second one is .85 • .15 = .1275. That would occur about 25 times. The probability that she misses the first shot and makes no points would be .15 • .00 = .00 which is likely to happen 30 times during the season.

Related Files:

www.ceismc.gatech.edu/ksi

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Further investigations:
Between video games, ask your student to describe or demonstrate how one of the characters moves across the screen. Then ask him to pretend the character is an algebraic function, \( f(x) \), and to represent the movement with a transformation.

When formulas appear in the news or on your job, discuss them with your student. Talk about when they work. In particular, ask your student to identify the domain and the range.

Terminology:
End behavior: A description of how a graph appears as its independent variable gets very large and very small.

Even function: A function that produces the same output for a given input and the opposite of that input. \( f(-x) = f(x) \).
Example: \( f(x) = x^2 \) then \( f(-x) = x^2 \)

Extraneous solution: A solution of a simplified version of an equation that does not satisfy the original equation.

Factor: To determine two or more expressions whose product is equivalent to a given expression.

Leading coefficient: In a polynomial, the number multiplied by the variable term with the highest degree.

Odd function: A function that produces the opposite output for the opposite input. \( f(-x) = -f(x) \).
Example: \( f(x) = x^3 \) then \( f(-x) = -x^3 \)

Quadratic expression: A polynomial whose highest degree is 2.
Example: \( x^2 + 3x - 5 \)

Transformations: Operations that alter the form of a geometric figure. Transformations in Math I include translations (shifts), dilations (stretches and shrinks), rotations (turns), and reflections (flips).

Zero factor property: When the product of two or more factors is 0, at least one of the factors is 0.

Zero of a function: A value for \( x \) that makes \( f(x) \) equal to 0.

Clues: Many questions are answered by writing and solving equations. This unit looks at various techniques for solving non-linear equations, including graphically. The unit also extends the study of basic mathematical functions. When functions are transformed, they provide better fits for more phenomena.

Related Files: www.ceismc.gatech.edu/csi

Algebraic Investigations

Students will:

- Graph transformations of basic functions including vertical shifts, stretches and shrinks, and reflections across the \( x \)- and \( y \)-axes.
- Explain and interpret characteristics of functions: domain, range, zeros, intercepts, intervals of increase and of decrease, maximum and minimum values, and end behavior.
- Determine whether a function has symmetry and whether it is odd, even, or neither.
- Simplify, factor, and operate with radical, polynomial, and rational expressions.
- Solve simple quadratic, square root, and rational equations algebraically and graphically.

Classroom Cases:

1. Steve has joined a gym and is exercising to reduce his percentage of body fat. A trainer at the gym measured skin folds at Steve’s chest, abdomen, and thigh. The sum (s) of these measurements is 73. Body density (d) can be predicted using this measurement, and percentage of body fat (f) can be calculated using body density.
   \[
   d(s) = 1.109380 - 0.0008267s + 0.0000016s^2 - 0.0002574 \times age
   \]
   \[
   f(d) = ((4.95/d) - 4.50)100
   \]
   a. If Steve is 18 years old, what is his percentage of body fat now?
   b. If he wants to reduce his body fat percentage to 15, what should his body density be?
   What should the sum of the skin fold measurements be?

   Case Closed - Evidence:
   a. \( d(73) = 1.109380 - 0.0008267 \times 73 + 0.0000016 \times 73^2 - 0.0002574 \times 18 = 1.0529 \)
   \( f(1.0529) = ((4.95/1.0529) - 4.50)100 = 20.119 \) About 20% of Steve’s body is fat.

   b. \( 15 = f(d) = ((4.95/d) - 4.50)100 \)
   \( d = 1.06451 \). I entered \( y_1 = 1.109380 - 0.0008267x + 0.0000016x^2 - 0.0002574x \times 18 \) and \( y_2 = 1.06451 \) in my calculator. I found the intersection of these graphs, and the domain value of that point, 54.399, is the sum of the three skin folds that will indicate 15% body fat.

2. Your friend, Ava Walker, wants a large monogram on her towels, a medium one on her book bag, and small ones on the collars of her blouses. Design a monogram and define it mathematically (function rules and domains).

   Case Closed - Evidence:

3. Solve the following equations for \( x \) where \( x \) is a Real number:
   a. \( x^2 - 14x = -49 \)
   b. \( 4 + \sqrt{x} = 3 \)
   c. \( x^2 - 8 = 0 \)

   Case Closed - Evidence:
   a. \( x^2 - 14x + 49 = 0 \)
   b. \( 4 + \sqrt{x} = 3 \)
   c. \( x^2 - 8 = 0 \)
   \( x = 7 \)
   \( x = 1, \) but \( 4 + \sqrt{1} \neq 3 \)
   1 is extraneous; there is no Real number solution.
**Further investigations:**
Ask your student to compare and contrast map coordinates to the standard coordinate grid. Let him explain the similarities and differences.

Question your student about computer screen coordinates and the standard coordinate grid. How are they alike? How are they different?

The coordinate grid system is named after Rene Descartes. Encourage your student to research the contributions Descartes made to mathematics.

**Terminology:**

**Concave polygon:** A polygon with one or more diagonals that have points outside the polygon.

**Convex polygon:** A polygon with all interior angles measuring less than 180°. All diagonals of a convex polygon are inside the polygon.

**Distance formula:** The equation to find the length between two points on the coordinate plane.

**Midpoint:** The point on a line segment that divides it into two equal parts.

**Non-collinear:** Not on the same line

**Clues:**
One of the really big and powerful ideas in mathematics is coordinate geometry. In coordinate geometry, we identify points in a plane with pairs of numbers. In fact, we do this all the time. For instance, I might say that I live 3 miles east and ½ mile south of Georgia Tech. The great power of coordinate geometry is that it allows us to solve geometry problems by algebra, and to solve algebra problems by geometry.

**Related Files:**
www.ceismc.gatech.edu/csi

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**Coordinate Geometry**

**Students will:**
- Investigate properties of geometric figures in the coordinate plane
- Determine the distance between two points in the plane
- Determine the distance between a point and a line in the plane
- Determine the midpoint of a line segment
- Use the coordinate plane to investigate properties of and verify conjectures related to triangles and quadrilaterals

**Classroom Cases:**

1. **Quadrilateral Investigation**
   a. Place 4 non-collinear points on your graph paper. Write the coordinates next to each point.
   b. Connect the points, making a quadrilateral—it may be convex or concave.
   c. Find the midpoint of each side of the quadrilateral and indicate the coordinates. Join the midpoints, forming a second quadrilateral.
   d. Use slope, length (distance formula), and any other information you deem useful to classify the second quadrilateral
   e. Will the second quadrilateral always be of this type? How can you test and prove or disprove your conjecture?

**Case Closed - Evidence:**

<table>
<thead>
<tr>
<th>Point</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>(-1.00, 5.00)</td>
</tr>
<tr>
<td>Q</td>
<td>(-4.50, 2.00)</td>
</tr>
<tr>
<td>A</td>
<td>(2.00, -1.00)</td>
</tr>
<tr>
<td>D</td>
<td>(-1.50, -4.00)</td>
</tr>
</tbody>
</table>

The slopes of opposite sides are the same, therefore the figure is a parallelogram.

Slope of \( \overline{QU} = 0.86 \) and slope of \( \overline{AD} = 0.86 \)
Slope of \( \overline{DQ} = -2 \) and slope of \( \overline{UA} = -2 \)

2. **Triangle Investigation**
   a. Graph the following points: A (-1, -1), B (-3, 1), and C(1, 1).
   b. Identify what type of triangle you have graphed and justify your answer.

**Case Closed - Evidence:**

The distance from \( A \) to \( B \) is equal to the distance from \( A \) to \( C \). So sides \( \overline{AB} \) and \( \overline{AC} \) are congruent. Therefore the triangle is isosceles. Slope of \( \overline{AB} = -1 \) and slope of \( \overline{AC} = 1 \). Since the slopes are negative reciprocals, the line segments (sides) are perpendicular and \( \angle A \) is a right angle. Therefore, \( \triangle ABC \) is a right isosceles triangle.